Reduction of amplitude equations by the renormalization group approach

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This article elucidates and analyzes the fundamental underlying structure of the renormalization group (RG) approach as it applies to the solution of any differential equation involving multiple scales. The amplitude equation derived through the elimination of secular terms arising from a naive perturbation expansion of the solution to these equations by the RG approach [L.-Y. Chen, N. Goldenfeld, and Y. Oono, Phys. Rev. E **54**, 376 (1996)] is reduced to an algebraic equation which is expressed in terms of the *Thiele semi-invariants or cumulants* of the eliminant sequence $\{Z_i\}_{i=1}^{\infty}$. Its use is illustrated through the solution of both linear and nonlinear perturbation problems and certain results from the literature are recovered as special cases. The fundamental structure that emerges from the application of the RG approach is not the amplitude equation but the aforementioned algebraic equation.

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I. INTRODUCTION

The method of renormalization group (RG) [1] as a means of eliminating secular terms that arise in the naive expansion of the solution of differential equations has had remarkable success during the past decade in generating asymptotic solutions to problems that were unwieldy to integration to higher orders and which had previously been attacked by methods that fit the particular problems, such as the method of multiple scales for oscillators, matched asymptotic expansions for boundary layer problems, etc. The development of the RG method continued in Ref. [2] by introducing an approach based on the classical theory of envelopes to setting its mathematical foundations. Many authors have attempted to provide additional explanations for the mathematical framework underlying the technique. Its relation to the method of averaging was elucidated in a series of papers [3,4]. Lie-group approaches were developed [5] to simplify the process of deriving asymptotic solutions employing translational symmetry generators, and later applied to pulse dynamics [6]. An interesting approach was introduced in [7,8] to simplify the derivation of amplitude equations. Despite the merit of these later efforts, the RG method has not reached a degree of simplicity that will make it approachable and accessible to wider audiences, while the mechanism behind its spectacular successes has not been elucidated yet.

In this paper, departing from the developments in [1,7], we show that the amplitude equations derived through the standard RG approach can actually be integrated and thus provide an algebraic relation that, for a nonlinear problem, defines the amplitude implicitly, while for a linear problem leads to the actual asymptotic solution, *circumventing the necessity of performing the process of renormalization*. Furthermore, no further integrations are required. This finding is an important step in the demystification of the method and hopefully will provide a suitable impetus for the application of the method to other areas of statistical mechanics and field theory that use perturbative expansions but suffer from the presence of nonuniformities.

In Sec. II A we introduce the reader to the standard method of renormalization group with a most simple example: To facilitate the transition into the formalism introduced in this paper. In Sec. II B we introduce the main equations of this paper and show that the eliminant sequence $\{Z_i\}_{i=1}^{\infty}$ and the integral of the RG equation bear the same relationship as the moments have with the cumulants or *Thiele semi-invariants* [9,10] in statistical theories. Mere knowledge of the eliminant sequence provides the algebraic relation *and* the RG (amplitude) equation without any further effort. This is a general characteristic of the RG approach, and applies to all types of differential equations. Furthermore, we emphasize the fact that, from a differential equations point of view, the fundamental quantity in any RG analysis must be the algebraic relation and not the amplitude equation (a differential equation). The latter is just a means of solving the former, invoking the implicit function theorem.

In Sec. III, a further simplification of the process of deriving the asymptotic solution by the RG method is achieved for the case of linear equations. The asymptotic solution can be expressed in terms of the cumulants of the secular sequence $\{y_{ip}\}_{i=1}^{\infty}$. Mere knowledge of the latter provides the asymptotic solution without renormalizing the constants or integrating an amplitude equation. The algebraic relation here is linear in the amplitude, thus providing the justification for this simplification. We illustrate the application of the method with a few examples, of ascending difficulty, starting, for pedagogical purposes, with a linear oscillator with constant coefficients, continuing with linear oscillators with variable coefficients, linear eigenvalue problems, such as the linear anharmonic oscillator [11,12], and a boundary-layer problem.

In Sec. IV, the formalism is applied to the case of nonlinear differential equations such as the Rayleigh and Duffing equations. In the process we see that the aforementioned implicit relation defining the amplitude provides its phase without any extra effort.

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II. DERIVATION OF THE FIRST INTEGRAL OF THE AMPLITUDE EQUATION

The results to derive in this section are general and apply to both linear and nonlinear ordinary differential equations characterized by a perturbing parameter ϵ. In order to motivate the discussion and provide access to a wider array of audiences, we demonstrate the method by use of a linear oscillator problem. Standard nonlinear problems that have been analyzed in detail with the standard method of renormalization group [1,7] can be treated with this formalism and appear in Sec. IV.

A. Analysis of a problem with the standard RG method

Consider the second order linear ordinary differential equation

$$\ddot{y} - \epsilon \dot{y} + y = 0, \quad \epsilon \to 0+ . \tag{1}$$

The negative damping has been chosen intentionally. A naive perturbation expansion

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots, \tag{2}$$

leads to the following hierarchy of equations:

$$\ddot{y}_0 + y_0 = 0, \tag{3}$$

$$\ddot{y}_n + y_n = \dot{y}_{n-1}, \quad n \geq 1. \tag{4}$$

The solutions of the above equations are

$$y_0 = Ae^{it} + A^* e^{-it}, \tag{5}$$

which is the solution whose constants A, A^* we will eventually renormalize, and

$$y_1 = \frac{1}{2} A t e^{it} + \text{c.c.},$$

$$y_2 = \frac{1}{8} A(t^2 - it)e^{it} + \text{c.c.},$$

$$y_3 = \frac{1}{16} A\left(\frac{t^3}{3} - it^2\right)e^{it} + \text{c.c.},$$

$$y_4 = \frac{1}{64} A\left(\frac{t^4}{6} - it^3 - \frac{t^2}{2} - \frac{i}{2}\right)e^{it} + \text{c.c.}, \tag{6}$$

are the particular solutions of the higher order equations. Thus

$$y_{1p} = \frac{1}{2}t, \quad y_{2p} = \frac{1}{8}(t^2 - it), \quad y_{3p} = \frac{1}{16}\left(\frac{t^3}{3} - it^2\right), \ \ldots , \tag{7}$$

where A is a *constant* complex amplitude. We call the sequence $\{y_{ip}\}_{i=1}^{\infty}$ the "secular sequence" and the series $y_p = [1 + \epsilon y_{1p} + \epsilon^2 y_{2p} + O(\epsilon^3)]$ the "secular series." In the standard RG approach [1,7], the time variable appearing in each term of the secular series is split as $t = (t-\tau) + \tau$, $t^2 = (t^2 - \tau^2) + \tau^2, \ldots$ and then the terms $\tau, \tau^2, \ldots$ are absorbed into the renormalization constant Z of the free parameter A in the sense that $A = \mathcal{A}(\tau)Z$ or explicitly in the form of a slowly varying amplitude

$$A = \mathcal{A}(\tau)[1 + \epsilon Z_1(\tau, \mathcal{A}) + \epsilon^2 Z_2(\tau, \mathcal{A}) + O(\epsilon^3)], \tag{8}$$

where we call $\{Z_i\}_{i=1}^{\infty}$ the "eliminant sequence." One then substitutes the above form for the free parameter A into the particular solutions (6) and determines the Z_i arising in each order in ϵ of the zeroth order solution in Eq. (5), i.e.,

$$y_0 = \mathcal{A}(\tau)[1 + \epsilon Z_1(\tau, \mathcal{A}) + \epsilon^2 Z_2(\tau, \mathcal{A}) + O(\epsilon^3)]e^{it} + \text{c.c.}, \tag{9}$$

by comparing with the corresponding expressions one obtains when the expanded form of A is substituted into the higher order particular solutions of Eq. (6). For example, it is straightforward to derive the first three members of the eliminant sequence as

$$Z_1 = -\frac{\tau}{2}, \quad Z_2 = \frac{1}{8}\tau^2 + \frac{i}{8}\tau, \quad Z_3 = -\frac{1}{16}\left(\frac{\tau^3}{3} + i\tau^2\right). \tag{10}$$

The amplitude equation is obtained by differentiating Eq. (8) with respect to the parameter τ [1,7], and equating τ with time t

$$\frac{d\mathcal{A}}{dt} = -\mathcal{A}\epsilon\left[\frac{dZ_1}{dt} + \epsilon\left(\frac{dZ_2}{dt} - Z_1\frac{dZ_1}{dt}\right)\right] + O(\epsilon^3). \tag{11}$$

Performing the required differentiations of the terms in the eliminant sequence and substituting into Eq. (11), we obtain

$$\frac{d\mathcal{A}}{d\tau} = -\epsilon\mathcal{A}\left(-\frac{1}{2} + \epsilon\frac{i}{8}\right) + O(\epsilon^4). \tag{12}$$

This is the celebrated amplitude equation associated with the oscillator (1). Despite the significance of the above result, its derivation relied on secondary parameters such as τ and required a process of elimination that, although unambiguous, becomes unwieldy in higher orders of approximation. Furthermore, despite its important role in extracting global features, it is not immediately obvious how to integrate the amplitude equation (especially in the case of a nonlinear equation). These topics are discussed in the subsequent sections.

B. Reduction of the amplitude equation and main equations

In this section we show that the amplitude Eq. (11) possesses a first integral. For nonlinear equations in some cases (but not all) the complete solution is provided. This is the case when this method is applied to the Duffing equation. In the next section we also show that, for linear equations, the process of renormalization is not required to derive the amplitude equation or its solution. Both results are based in the construction that follows.

As discussed above briefly, the standard RG approach [1,7] commences by introducing a near-identity transformation in the form of a slowly varying amplitude (8) that will subsequently absorb any secular terms that may appear in the asymptotic expansion of the solution. The amplitude equation is obtained by noting that A is a constant and thus differentiating with respect to the fictitious parameter τ and equating with the time t [1] or, equivalently, eliminating the parameter τ characterizing the family of asymptotic solutions in favor of their envelope [2] leads to the following differential equation for the slowly varying amplitude $\mathcal{A}$ which, in an expanded form, reads

$$\frac{d\mathcal{A}}{dt} = -\mathcal{A}\epsilon\Bigg[\frac{dZ_1}{dt} + \epsilon\left(\frac{dZ_2}{dt} - Z_1\frac{dZ_1}{dt}\right) + \epsilon^2\left(\frac{dZ_3}{dt} - Z_1\frac{dZ_2}{dt} - Z_2\frac{dZ_1}{dt} + Z_1^2\frac{dZ_1}{dt}\right) + \epsilon^3\left(\frac{dZ_4}{dt} + (2Z_1Z_2 - Z_3 - Z_1^3)\frac{dZ_1}{dt} + \frac{dZ_2}{dt}(Z_1^2 - Z_2) - Z_1\frac{dZ_3}{dt}\right)\Bigg] + O(\epsilon^5). \tag{13}$$

However, it is apparent that the above equation has a first integral which obtains the form

$$\ln\mathcal{A} = \ln\mathcal{A}(0) - \epsilon Z_1 - \epsilon^2\left(Z_2 - \frac{Z_1^2}{2}\right) - \epsilon^3\left(Z_3 - Z_1Z_2 + \frac{Z_1^3}{3}\right) - \epsilon^4\left(Z_4 - Z_1Z_3 - \frac{Z_2^2}{2} + Z_1^2Z_2 - \frac{Z_1^4}{4}\right) + O(\epsilon^5). \tag{14}$$

The various terms multiplying the powers of ϵ are the *Thiele semi-invariants* [9,10] or *cumulants* of the sequence $\{Z_i\}_{i=1}^{\infty}$ and their analytical form *to any order* is tabulated in the Appendix.

Relation (14) is a consequence of the structure of the *standard* RG method [1] and its derivation relies on secondary parameters. However, the result (14) can be derived without resorting to these additional parameters. Instead, introducing a near-identity transformation of the form

$$A = \mathcal{A}(t)[1 + \epsilon Z_1(t,\mathcal{A}) + \epsilon^2 Z_2(t,\mathcal{A}) + O(\epsilon^3)] \tag{15}$$

and determining the sequence $\{Z_i\}_{i=1}^{\infty}$ through standard renormalization, we proceed by taking the natural logarithm of both sides of the above expression while simultaneously setting $A=\mathcal{A}(0)$, since A is a constant. This process immediately leads to the fundamental result (14) directly from the definition of the cumulants in the Appendix.

The amplitude equation (13) is just a means of calculating the amplitude that arises in the implicit Eq. (14). The algebraic relation (14) is the fundamental outcome of the RG approach to differential equations. More precisely, to solve for $\mathcal{A}(t)$ in Eq. (14), one resorts to the implicit function theorem. If the conditions of the theorem are satisfied, then implicit differentiation can be applied that leads to Eq. (13) and as a consequence to an explicit form for $\mathcal{A}(t)$.

C. Application to the linear oscillator

The application of Eq. (14) to the linear oscillator of Sec. II A is straightforward. Knowledge of the eliminant sequence $\{Z_i\}_{i=1}^{\infty}$ calculated in Eq. (10) leads to the solution of the amplitude equation in the form

$$\ln\mathcal{A} = \ln\mathcal{A}(0) + \frac{\epsilon t}{2} - i\frac{\epsilon^2}{8}t - i\frac{\epsilon^4}{128}t + O(\epsilon^5)$$

$$\text{or}\quad \mathcal{A} = \mathcal{A}(0)e^{\epsilon t/2 + i(-\epsilon^2/8 - \epsilon^4/128)t + O(\epsilon^5)}. \tag{16}$$

The amplitude equation can be recovered from the first of the two above expressions. Application of relation (14) to nonlinear equations is taken over in Sec. IV. As far as linear equations are concerned, both the amplitude equation and its first integral can be determined by mere knowledge of the secular sequence $\{y_{ip}\}_{i=1}^{\infty}$ circumventing the need to perform renormalization. This route is taken over in the next section.

III. SOLUTION OF LINEAR EQUATIONS BY MEANS OF AN IMPROVEMENT OF ALGEBRAIC RELATION (14)

For a linear differential equation, there is a certain relationship between the secular sequence $\{y_{ip}\}_{i=1}^{\infty}$ and the eliminant sequence $\{Z_i\}_{i=1}^{\infty}$,

$$Z_1 = (-1)y_{1p},$$

$$Z_2 = (-1)^2y_{1p}^2 - y_{2p},$$

$$Z_3 = (-1)^3y_{1p}^3 + 2(-1)^2y_{1p}y_{2p} - y_{3p},$$

$$\vdots = \vdots\,,$$

whereby their respective cumulants are also related:

$$Z_1 = -y_{1p},$$

$$Z_2 - \frac{Z_1^2}{2} = -\left(y_{2p} - \frac{y_{1p}^2}{2}\right),$$

$$Z_3 - Z_1Z_2 + \frac{Z_1^3}{3} = -\left(y_{3p} - y_{1p}y_{2p} + \frac{y_{1p}^3}{3}\right), \tag{17}$$

$$\vdots = \vdots\,. \tag{18}$$

Thus the algebraic relation (14) characterizing $\mathcal{A}$ obtains the form

$$\ln\mathcal{A} = \ln\mathcal{A}(0) + \epsilon y_{1p} + \epsilon^2\left(y_{2p} - \frac{y_{1p}^2}{2}\right) + \epsilon^3\left(y_{3p} - y_{1p}y_{2p} + \frac{y_{1p}^3}{3}\right) + \epsilon^4\left(y_{4p} - y_{1p}y_{3p} - \frac{y_{2p}^2}{2} + y_{1p}^2y_{2p} - \frac{y_{1p}^4}{4}\right) + O(\epsilon^5). \tag{19}$$

The solution in this case is automatically found by mere knowledge of the secular polynomials $\{y_{ip}\}_{i=1}^{\infty}$, without resorting to the amplitude equation or performing renormalization. Relation (19) provides the *RG amplitude to any linear differential equation*. We illustrate these results with a sequence of problems of ascending difficulty.

A. Linear oscillator revisited

To determine the asymptotic solution of Eq. (1), we only need knowledge of the secular sequence $\{y_{ip}\}_{i=1}^{\infty}$. From Sec. II A, its first three members are

$$y_{1p}=\frac{1}{2}t, \quad y_{2p}=\frac{1}{8}(t^2-it), \quad y_{3p}=\frac{1}{16}\left(\frac{t^3}{3}-it^2\right), \ldots .$$

Calculating the cumulants

$$\kappa_1 \equiv y_{1p}=\frac{1}{2}t, \quad \kappa_2 \equiv y_{2p}-\frac{y_{1p}^2}{2}=-\frac{1}{8}it,$$

$$\kappa_3 \equiv y_{3p}-y_{1p}y_{2p}+\frac{y_{1p}^3}{3}=0,\ldots \tag{20}$$

of the secular sequence and substituting into the algebraic expression (19) leads to the first integral of the amplitude equation in the form (16). Thus it was obtained without us performing the process of renormalization.

For completeness, we calculate the asymptotic solution to Eq. (1). The general (asymptotic) solution $y(t;\epsilon)=\mathcal{A}e^{it}+\mathcal{A}^*e^{-it}$ of Eq. (1) becomes

$$y(t;\epsilon)=\mathcal{A}(0)e^{\epsilon t/2+i(1-\epsilon^2/8-\epsilon^4/128)t}+\mathcal{A}^*(0)e^{\epsilon t/2-i(1-\epsilon^2/8-\epsilon^4/128)t}. \tag{21}$$

One can compare the RG solution (21) with its closed form counterpart

$$y(t)=e^{\epsilon t/2}(Ce^{it/2\sqrt{4-\epsilon^2}}+C^*e^{-it/2\sqrt{4-\epsilon^2}}). \tag{22}$$

For small ϵ, $\sqrt{4-\epsilon^2}=2\left[1-\frac{1}{2}(\epsilon/2)^2-\frac{1}{8}(\epsilon/2)^4+O(\epsilon^6)\right]$. Substituting into Eq. (22) we exactly recover the renormalization group solution (21).

B. Linear equations with variable coefficients

Consider the linear oscillator with forcing $k=1-\epsilon t$. If $k>0$, $\epsilon t<1$, the forcing is restoring and brings the system back to equilibrium. Otherwise, when $k<0$, $\epsilon t>1$, the forcing becomes increasingly repelling. This behavior should be captured in the asymptotic solution of the problem

$$\ddot{y}+y=\epsilon t y, \quad y(0)=1, \quad y'(0)=0. \tag{23}$$

A power series expansion of the solution in terms of ϵ leads to the hierarchy of equations

$$\ddot{y}_0+y_0=0, \tag{24}$$

$$\ddot{y}_n+y_n=ty_{n-1}, \quad n\geq 1. \tag{25}$$

The corresponding solutions are

$$y_0=Ae^{it}+A^*e^{-it}, \tag{26}$$

$$y_1=-\frac{1}{4}A(it^2-t)e^{it}+\text{c.c.}, \tag{27}$$

$$y_2=A\left(-\frac{1}{32}t^4-i\frac{5}{48}t^3+\frac{5}{32}t^2+\frac{5i}{32}t\right)e^{it}+\text{c.c.} \tag{28}$$

After a straightforward calculation of the cumulants of the sequence $\{y_{ip}\}_{i=1}^{\infty}$, the algebraic expression for the first integral (19) reads

$$\ln\mathcal{A}=\ln\mathcal{A}(0)+\epsilon\frac{1}{4}(-it^2+t)+\epsilon^2\left[t^2/8+i\left(\frac{5t}{32}-\frac{t^3}{24}\right)\right]$$
$$+O(\epsilon^3), \tag{29}$$

and the renormalization group solution of the differential equation in Eq. (23)

$$y(t;\epsilon)=e^{\epsilon t/4+\epsilon^2t^2/8}[\mathcal{A}(0)e^{i[t-\epsilon t^2/4+\epsilon^2(5t/32-t^3/24)]}$$
$$+\mathcal{A}^*(0)e^{-i[t-\epsilon t^2/4+\epsilon^2(5t/32-t^3/24)]}]. \tag{30}$$

Employing the initial conditions, leads to

$$\mathcal{A}(0)=\frac{1}{2}+i\frac{\epsilon/8}{1+\epsilon^2\frac{5}{32}}, \tag{31}$$

and the solution of the initial value problem (23) in the form

$$y(t;\epsilon)=e^{\epsilon t/4+\epsilon^2t^2/8}\left(\cos\omega(t;\epsilon)t-\frac{\epsilon/4}{1+\epsilon^2\frac{5}{32}}\sin\omega(t;\epsilon)t\right), \tag{32}$$

where

$$\omega(t;\epsilon)=1-\epsilon\frac{t}{4}+\epsilon^2\left(\frac{5}{32}-\frac{t^2}{24}\right). \tag{33}$$

We note that the above solution agrees with the one derived in [1], if we only retain terms of first order in ϵ. Second, the unboundedness of the solution, due to the repelling force, is reflected in the leading exponent in Eq. (32) when $\epsilon t>1$.

C. Linear eigenvalue problems: The quantum mechanical anharmonic oscillator

We consider the time-independent Schrödinger equation

$$\left(-\frac{d^2}{dx^2}+\frac{1}{4}x^2+\frac{1}{4}\epsilon x^4-E(\epsilon)\right)\psi(x)=0 \tag{34}$$

for the ground-state wave function $\psi(x)$, subject to $\psi(x=\pm\infty)=0$. Normally, the above problem is solved with conventional weak-coupling Rayleigh-Schrödinger perturbation theory, representing the eigenfunction and eigenvalue as a power series in ϵ

$$\psi(x)=\sum_{n=0}^{\infty}\epsilon^n y_n(x), \quad E(\epsilon)=\sum_{n=0}^{\infty}\epsilon^n E_n. \tag{35}$$

The above is an asymptotic series. In [11], Bender and Wu showed that the solution consists of the ground-state wave function of the harmonic oscillator as a zeroth order approximation, $y_0=Ae^{-x^2/4}$, $E_0=1/2$, while the higher order approximations are expressed through a sequence of polynomials $P_n(x)$ that satisfy the recursion formula

$$\frac{d^2P_n(x)}{dx^2} - x\frac{dP_n(x)}{dx} = \frac{1}{4}x^4P_{n-1}(x) - \sum_{j=0}^{n-1} P_j(x)E_{n-j}. \quad (36)$$

These polynomials can be expressed in closed form as

$$P_n(x) = \sum_{k=1}^{2n} C_{n,k}\left(-\frac{1}{2}x^2\right)^k, \quad (37)$$

where $P_0=1$, while the eigenvalues and the constants $C_{n,k}$ are expressed through another recursion relation. The first few higher order polynomials read [12]

$$P_1(x) = -\frac{3}{2}\left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^4,$$

$$P_2(x) = -\frac{21}{4}\left(\frac{x}{2}\right)^2 + \frac{31}{8}\left(\frac{x}{2}\right)^4 + \frac{13}{6}\left(\frac{x}{2}\right)^6 + \frac{1}{2}\left(\frac{x}{2}\right)^8, \ldots, \quad (38)$$

while the corresponding cumulants are [13]

$$\kappa_1 = P_1(x) = -\frac{3}{2}\left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^4,$$

$$\kappa_2 = P_2(x) - P_1^2(x)/2 = \frac{21}{4}\left(\frac{x}{2}\right)^2 + \frac{11}{4}\left(\frac{x}{2}\right)^4 + \frac{1}{3}\left(\frac{x}{2}\right)^6, \ldots . \quad (39)$$

The sequence of polynomials $\{P_i\}_{i=1}^{\infty}$ corresponds to the secular sequence $\{y_{ip}\}_{i=1}^{\infty}$ in our notation that multiply the fundamental solution $e^{-x^2/4}$ (compare with the fundamental solution e^{it} of the oscillators in the previous two examples). As a consequence of the fact that the problem (34) is linear, the algebraic relation (19) leads to

$$\ln \mathcal{A} = \ln \mathcal{A}(0) + \epsilon P_1 + \epsilon^2(P_2 - P_1^2/2) + O(\epsilon^2), \quad (40)$$

the amplitude

$$\mathcal{A} = \mathcal{A}(0)e^{\epsilon P_1 + \epsilon^2(P_2 - P_1^2/2) + O(\epsilon^2)}, \quad (41)$$

and the asymptotic wave-function

$$\psi(x;\epsilon) = \mathcal{A}(0)\exp\left(-\frac{x^2}{4} + \sum_{n=1}^{\infty} \epsilon^n \kappa_n(x)\right), \quad (42)$$

where $\{\kappa_i\}_{i=1}^{\infty}$ is the sequence of cumulants, as these are defined in the Appendix.

We note that the above solution is not new. Kunihiro [13] derived this expression by means of a process of differentiation and elimination of parameters that arose in his scheme of generating the envelope of a family of (asymptotic) solutions to the Schrödinger equation. We demonstrated here that the appearance of the cumulants in [13] is a result of the structure of the RG approach, embodied in the algebraic relation (19), rather than being a special characteristic of the Bender-Wu perturbation theory.

D. Linear boundary-layer problems

The standard RG approach [1] introduces scaled variables prior to the solution of boundary layer problems. This was shown not be necessary in [7] if an alternative method is employed. Our construction will follow [1] by means of a dominant balance process.

Consider the linear boundary layer problem with constant coefficients

$$a\epsilon\frac{d^2y}{dt^2} + b\frac{dy}{dt} + y = 0. \quad (43)$$

We assume that $b/a > 0$. A dominant balance analysis shows the existence of a boundary layer of thickness ϵ near the origin. Introducing the inner variables $x=\epsilon X$, $y(x)=Y(X)$, we can express the above equation in the form

$$a\frac{d^2Y}{dX^2} + b\frac{dY}{dX} + \epsilon Y = 0. \quad (44)$$

Employing a naive asymptotic expansion of the solution generates the following hierarchy of equations

$$a\frac{d^2Y_0}{dX^2} + b\frac{dY_0}{dX} = 0, \quad (45)$$

$$a\frac{d^2Y_n}{dX^2} + b\frac{dY_n}{dX} = -Y_{n-1}, \quad n \geq 1, \quad (46)$$

and corresponding particular solutions

$$Y_0 = A + Be^{-(b/aX)},$$

$$Y_1 = \left(-\frac{1}{b}A + \frac{1}{b}Be^{-(b/aX)}\right)X,$$

$$Y_2 = A\left(\frac{X^2}{2b^2} - \frac{a}{b^3}X\right) + B\left(\frac{X^2}{2b^2} + \frac{a}{b^3}X\right)e^{-(b/aX)}. \quad (47)$$

For the part of the solution associated with the constants A and B, the expression for the first integral of the amplitude (19) leads to

$$\ln \mathcal{A} = \ln \mathcal{A}(0) + \epsilon Y_{1A} + \epsilon^2\left(Y_{2A} - \frac{Y_{1A}^2}{2}\right),$$

$$\ln \mathcal{B} = \ln \mathcal{B}(0) + \epsilon Y_{1B} + \epsilon^2\left(Y_{2B} - \frac{Y_{1B}^2}{2}\right),$$

where

$$Y_{1A} = -\tfrac{1}{b}X, \quad Y_{1B} = \tfrac{1}{b}X,$$

$$Y_{2A} = \tfrac{X^2}{2b^2} - \tfrac{a}{b^3}X, \quad Y_{2B} = \tfrac{X^2}{2b^2} + \tfrac{a}{b^3}X. \quad (48)$$

Substituting directly into the first integral (19) we obtain

$$\mathcal{A}(X) = \mathcal{A}(0)e^{-\epsilon X/b - \epsilon^2(a/b^3)X}, \quad \mathcal{B}(X) = \mathcal{B}(0)e^{-(b/aX) + \epsilon X/b + \epsilon^2(a/b^3)X}. \quad (49)$$

Thus the asymptotic solution to the problem in inner coordinates is

$$Y(X) = \mathcal{A}(0)e^{-\epsilon X/b-\epsilon^2(a/b^3)X} + \mathcal{B}(0)e^{\epsilon X/b+\epsilon^2(a/b^3)X}e^{-b/aX}. \tag{50}$$

The solution to the original problem (43) assumes the form

$$y(x) = \mathcal{A}(0)e^{-x/b-\epsilon ax/b^3} + \mathcal{B}(0)e^{(-1/\epsilon)(bx/a)+x/b+\epsilon ax/b^3}. \tag{51}$$

For completeness, one can compare the above asymptotic solution with its closed form counterpart

$$y(x) = C_1 e^{(-b+\sqrt{b^2-4a\epsilon}/2a\epsilon)x} + C_2 e^{(-b-\sqrt{b^2-4a\epsilon})/2a\epsilon x}. \tag{52}$$

Expanding the radical in powers of $4a\epsilon/b^2$, we recover the solution (51) when we include powers of ϵ up to order two.

IV. NONLINEAR EQUATIONS

A. Rayleigh equation

The Rayleigh equation introduced by J.W. Strutt as a mathematical model for certain problems in acoustics has been used as a benchmark problem to illustrate the method of multiple scales [14] as well as the method of renormalization group [1,7]. For completeness we formulate the problem and calculate the hierarchy of particular solutions y_i up to and including terms of order two in ϵ. The Rayleigh oscillator has the form

$$\ddot{y} + y = \epsilon\dot{y}\left(1 - \frac{1}{3}\dot{y}^2\right). \tag{53}$$

Expanding the solution y in a series in powers of ϵ, $y=y_0+\epsilon y_1+\epsilon^2 y_2+\cdots$ leads to the hierarchy of equations

$$\ddot{y}_0 + y_0 = 0, \tag{54}$$

$$\ddot{y}_1 + y_1 = \dot{y}_0 - \frac{1}{3}\dot{y}_0^3, \tag{55}$$

$$\ddot{y}_2 + y_2 = \dot{y}_1(1 - \dot{y}_0^2), \tag{56}$$

$$\vdots = \vdots . \tag{57}$$

We include here only particular solutions to various orders,

$$y_0 = Ae^{it} + \text{c.c.},$$

$$y_1 = \frac{1}{2}te^{it}A(1-|A|^2) - \frac{i}{24}A^3e^{3it} + \text{c.c.},$$

$$y_2 = \frac{1}{8}A(1-4|A|^2+3|A|^4)t^2e^{it} + \frac{1}{16}iA(|A|^4-2)te^{it} + \frac{1}{16}iA^3(|A|^2-1)te^{3it} + \frac{1}{64}A^3(3|A|^2-2)e^{3it} - \frac{1}{192}A^5e^{5it} + \text{c.c.},$$

where A is a *constant* complex amplitude. Renormalization leads to

$$Z_1 = -\frac{1}{2}t(1-|\mathcal{A}|^2), \tag{58}$$

$$Z_2 = \frac{1}{8}(1-4|\mathcal{A}|^2+3|\mathcal{A}|^4)t^2 + \frac{1}{16}i(2-|\mathcal{A}|^4)t. \tag{59}$$

The corresponding cumulants are

$$Z_1 = -\frac{1}{2}t(1-|\mathcal{A}|^2), \tag{60}$$

$$Z_2 - \frac{Z_1^2}{2} = -\frac{1}{4}|\mathcal{A}|^2(1-|\mathcal{A}|^2)t^2 + \frac{1}{16}i(2-|\mathcal{A}|^4)t. \tag{61}$$

Substituting into the algebraic expression (14) leads to

$$\ln\mathcal{A}(t) = \ln\mathcal{A}(0) + \epsilon\frac{1}{2}t(1-|\mathcal{A}|^2) + \epsilon^2\left(-\frac{1}{4}|\mathcal{A}|^2(1-|\mathcal{A}|^2)t^2 + \frac{1}{16}i(|\mathcal{A}|^4-2)t\right) + O(\epsilon^3). \tag{62}$$

The above expression leads to an effortless derivation of the phase of $\mathcal{A}$ as follows. Reverting to plane polar coordinates $\mathcal{A}=\mathcal{R}e^{i\vartheta}$, relation (62) becomes

$$\ln\mathcal{R}(t) = \ln\mathcal{R}(0) + \epsilon\frac{1}{2}t(1-\mathcal{R}^2) - \epsilon^2\frac{1}{4}\mathcal{R}^2(1-\mathcal{R}^2)t^2 + O(\epsilon^3), \tag{63}$$

$$\vartheta(t) = \vartheta(0) + \epsilon^2\frac{1}{16}(\mathcal{R}^4-2)t + O(\epsilon^3). \tag{64}$$

Equation (64) is the phase one would obtain from the amplitude equation derived for example in [7]. Note that Eq. (63) is an implicit nonlinear algebraic relation for the polar amplitude. $\mathcal{R}$ can be obtained either by iteration or by resorting to the implicit function theorem. Then, implicit differentiation can be applied to provide a corresponding amplitude equation

$$\frac{1}{\mathcal{R}}\frac{d\mathcal{R}}{dt} = \epsilon\frac{1}{2}(1-\mathcal{R}^2) + O(\epsilon^3). \tag{65}$$

This result agrees with the one derived through the standard RG method [1] or the method of multiple scales [14].

B. Duffing equation

This is the classical analog of the quantum mechanical anharmonic oscillator with cubic nonlinearity

$$\ddot{y} + y + \epsilon y^3 = 0. \tag{66}$$

We expand the solution in a power series of ϵ that leads to a hierarchy of equations

$$\ddot{y}_0 + y_0 = 0, \tag{67}$$

$$\ddot{y}_1 + y_1 = -y_0^3, \tag{68}$$

$$\ddot{y}_2 + y_2 = -3y_0^2y_1, \tag{69}$$

$$\ddot{y}_3 + y_3 = -3y_0(y_0y_2 + y_1^2), \tag{70}$$

$$\vdots = \vdots \tag{71}$$

and the corresponding particular solutions

$$y_0 = Ae^{it} + A^*e^{-it}, \tag{72}$$

$$y_1 = \frac{3}{2}iA|A|^2te^{it} + \frac{1}{8}A^3e^{3it} + \text{c.c.}, \tag{73}$$

$$y_2 = -\frac{9}{8}A|A|^4t^2e^{it} - \frac{15}{16}iA|A|^4te^{it} + \frac{9}{16}iA^3|A|^2te^{3it} - \frac{21}{64}A^3|A|^2e^{3it} + \frac{1}{64}A^5e^{5it}, \tag{74}$$

$$y_3 = \left(\frac{45}{32}A|A|^6t^2 + \frac{123}{128}iA|A|^6t - \frac{9}{16}iA|A|^6t^3\right)e^{it} + \left(-\frac{81}{64}A^3|A|^4t^2 - \frac{117}{64}iA^3|A|^4t + \frac{417}{512}A^3|A|^4\right)e^{3it} + \left(-\frac{43}{512}A^5|A|^2 + \frac{15}{128}iA^5|A|^2t\right)e^{5it} + \frac{1}{512}A^7e^{7it}. \tag{75}$$

After renormalization and calculation of the corresponding cumulants, Eq. (14) leads to

$$\ln \mathcal{A} = \ln \mathcal{A}(0) + it\left(\epsilon\frac{3}{2}|\mathcal{A}|^2 - \epsilon^2\frac{15}{16}|\mathcal{A}|^4 + \epsilon^3\frac{123}{128}|\mathcal{A}|^6\right) + O(\epsilon^4). \tag{76}$$

Reverting to plane-polar coordinates $\mathcal{A}=\mathcal{R}e^{i\vartheta}$ provides the following pair of algebraic equations for the polar amplitude and phase:

$$\ln \mathcal{R}(t) = \ln \mathcal{R}(0) + O(\epsilon^4), \tag{77}$$

$$\vartheta(t) = \vartheta(0) + \left(\epsilon\frac{3}{2}\mathcal{R}^2 - \epsilon^2\frac{15}{16}\mathcal{R}^4 + \epsilon^3\frac{123}{128}\mathcal{R}^6\right)t + O(\epsilon^4). \tag{78}$$

This demonstrates the constancy of the polar amplitude $\mathcal{R}$ and shows that the phase is obtained in an effortless way. The above results agree with those derived in [4].

V. CONCLUDING REMARKS

In this article we introduced an algebraic relation (14) that emerges naturally when one performs the standard renormalization process, without resorting to secondary parameters. This relation also forms a first integral of the amplitude Eq. (13) derived by means of the standard renormalization group approach [1] for the solution of differential equations involving multiple time scales. The algebraic relation (14) is expressed in terms of the *Thiele semi-invariants or cumulants* of the eliminant series $\{Z_i\}_{i=1}^{\infty}$. In the case of linear differential equations this expression simplifies significantly as the cumulants of the eliminant series $\{Z_i\}_{i=1}^{\infty}$ are related to the cumulants of the secular series $\{y_{ip}\}_{i=1}^{\infty}$ through Eq. (17), circumventing the need to perform renormalization. For nonlinear problems, Eq. (14) still provides a simple way to calculate the amplitude equation. In general, one still needs to resort to the solution of an amplitude equation. But this is a natural solution process of any implicit equation and hence of Eq. (14).

We note that the zeroth order solution of an nth order nonlinear equation depends on n integration constants, which upon renormalization will lead to n *coupled* algebraic equations of the form (14) or equivalently to n *coupled* amplitude equations.

Furthermore, it is needless to say that, for a linear equation, the naive expansion corresponds to the Born approximation, while the cumulant expansion corresponds to the Rytov approximation as these are applied to problems of wave propagation in deterministic and random media [15,16].

It is expected that the formalism inherent in the algebraic relation (14), perhaps in line with [17] in terms of differential operators, will lead to the integration of nonlinear partial differential equations.

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APPENDIX

When a function Z and its logarithm can be represented in terms of a power series in ϵ as

$$Z = \sum_{n=0}^{\infty} \epsilon^n \frac{\mu_n}{n!}, \quad \mu_0 = 1, \quad \ln Z = \sum_{n=1}^{\infty} \epsilon^n \frac{\kappa_n}{n!}, \tag{A1}$$

then the coefficients κ_n, known as Thiele *semi-invariants* or cumulants [9,10], are related to the coefficients μ_n by

$$\kappa_1 = \mu_1,$$

$$\kappa_2 = \mu_2 - \mu_1^2,$$

$$\kappa_3 = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3,$$

$$\kappa_4 = \mu_4 - 4\mu_1\mu_3 - 3\mu_2^2 + 12\mu_1^2\mu_2 - 6\mu_1^4,$$

$$\vdots = \vdots$$

or in general

$$\kappa_j = \sum_{n_i} (-1)^{\sum n_i - 1}\left(\sum n_i - 1\right)! \prod_i \left(\frac{(\mu_{n_i}/i!)}{n_i!}\right), \tag{A2}$$

where the summation of products is over all sets of integers that satisfy $\sum_i in_i = j$.

Because the representation we introduced in the text for the secular series y_p does not divide each term by a suitable factorial, our result can be recovered from the above with the stipulation $\mu_n = y_{np}/n!$, $n=0,1,\ldots$ or $\mu_n = Z_{np}/n!$, $n=0,1,\ldots$ depending on the context. Equivalently, and referring to the notation in the text,

$$\kappa_n = \frac{1}{n!}\left(\frac{\partial^n \ln y_p}{\partial \epsilon^n}\right)\Bigg|_{\epsilon=0}, \quad n=1,2,\ldots, \tag{A3}$$

and $y_p = 1 + \epsilon y_{1p} + \epsilon^2 y_{2p} + \cdots$ is the series of secular terms, with a corresponding expression for the eliminant series.

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